Transmutations of Knowledge Systems

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Abstract

Within the AGM paradigm revision and contraction operators are constrained by a set of rationality postulates. The logical properties of a set of knowledge are not strong enough to uniquely determine a revision or contraction operation, therefore constructions for these operators rely on some form of underlying preference relation, such as a system of spheres, or an epistemic entrenchment ordering. The problem of iterated revision is determining how the underlying preference relation should change in response to the acceptance or contraction of information. We call this process a transmutation. Generalizing Spohn’s approach we define a transmutation of a well-ordered system of spheres using ordinal conditional functions. Similarly, we define the transmutation of a well-ordered epistemic entrenchment using ordinal epistemic entrenchment functions. We provide several conditions which capture the relationship between an ordinal conditional function and an ordinal epistemic entrenchment function, and their corresponding transmutations. These conditions allow an ordinal epistemic entrenchment function to be explicitly constructed from an ordinal conditional function, and vice versa, in such a way that the epistemic state and its dynamic properties are preserved.

1 INTRODUCTION

The AGM paradigm is a formal approach to theory change. The logical properties of a theory are not strong enough to uniquely determine a contraction, or revision operation, therefore the constructions for these operators rely on some form of underlying preference relation, such as a system of spheres [6], a nice preorder on models [11], or an epistemic entrenchment ordering [4]. Theory change operators based on such preference relations require the relation to be predetermined or given at the outset. Although this provides desirable theoretical freedom, it leads to difficulties in designing computer-based knowledge revision systems. In order to accommodate a subsequent theory change such a system would require guidance in determining a subsequent preference relation. Revision and contraction operators result in a theory or set of knowledge and not a modified preference ordering. Hence, the problem of iterated revision is determining how the underlying preference relation should change in response to the acceptance or contraction of information. According to Schlecta [15] “...iterated revision ... is a very common phenomenon for cognitive systems”. We refer to the process of changing the underlying preference relation, and hence the knowledge system as a transmutation.

We make use of ordinal conditional functions [18], such functions can be thought of as well-ordered system of spheres, with possibly empty partitions. A conditionalization [18, 2] is a specific constructive method for modifying ordinal conditional functions in such a way as to accommodate a revision (or a contraction), and results in another ordinal conditional function. We address the problem of iterated revision by generalizing Spohn’s conditionalization, in particular we define transmutations to be any modification of an ordinal conditional function, such...
that it satisfies the revision and contraction postulates, and results in another ordinal conditional function. We show that conditionalization is a transmutation.

We introduce an ordinal epistemic entrenchment function. We define transmutations for these structures, and provide perspicuous conditions which capture the relationship between ordinal conditional functions, and ordinal epistemic entrenchment functions, such that the same knowledge set, and transmutations are obtained.

We briefly describe the AGM paradigm in Section 2. Spohn’s approach is outlined and generalized in Sections 3, in particular ordinal conditional functions, together with their contraction, revision, and transmutation are described. In Section 4, we introduce ordinal epistemic entrenchment functions, and their contraction, revision, and transmutation. In Section 5 we provide conditions which capture the relationship between ordinal conditional functions and ordinal epistemic entrenchment functions, so that contractions, revisions, and certain transmutations are equivalent. These conditions provide explicit translations which allow an ordinal conditional function and an ordinal epistemic function to be constructed from one another in such a way that a knowledge system and its dynamic properties are preserved. Related work is described in Section 6, and a summary of our results is given in Section 7. Examples and proofs can be found in [22].

2 THE AGM PARADIGM

We begin with some technical preliminaries. Let $L$ denote a countable language which contains a complete set of Boolean connectives. We will denote formulae in $L$ by lower case Greek letters. We assume $L$ is governed by a logic that is identified with its set of Boolean connectives. We will denote formulae by lower case Greek letters. We assume

\( (d) \) if $\alpha$ is a truth-functional tautology, then $\vdash \alpha$.

\( (b) \) if $\alpha \vdash \beta$ and $\vdash \alpha$, then $\vdash \beta$ (modus ponens).

\( (c) \) $\vdash$ is consistent, that is, $\forall \bot$, where $\bot$ denotes the inconsistent theory.

\( (d) \) $\vdash$ satisfies the deduction theorem.

\( (e) \) $\vdash$ is compact.

The set of all logical consequences of a set $T \subseteq L$, that is $\{ \alpha : T \vdash \alpha \}$, is denoted by $\text{Cn}(T)$. The set of tautologies, $\{ \alpha : \vdash \alpha \}$, is denoted by $\top$, and those formulae not in $\top$ are referred to as nontautological. A theory of $L$ is any subset of $L$, closed under $\vdash$. A consistent theory of $L$ is any theory of $L$ that does not contain both $\alpha$ and $\neg \alpha$, for any formula $\alpha$ of $L$. A complete theory of $L$ is any theory of $L$ such that for any formula $\alpha$ of $L$, the theory contains $\alpha$ or $\neg \alpha$. A theory is finite if the consequence relation $\vdash$ partitions its elements into a finite number of equivalence classes. The dual atoms for a finite theory $T$ are those nontautological elements $\alpha \in T$ such that for all $\beta \in \text{Cn}(\{ \alpha \})$, either $\vdash \beta$ or $\vdash \alpha \equiv \beta$ [12]. We define $L^\infty$ to be the set of consistent nontautological formulae in $L$.

We introduce the following notation: $K_L$ is the set of all theories of $L$, and $\Theta_L$ is the set of all consistent complete theories of $L$. If $\alpha$ is a formula of $L$, define $[\alpha]$ to be the set of all consistent complete theories of $L$ containing $\alpha$. If $\alpha$ is inconsistent, then $[\alpha] = \emptyset$, and if $\vdash \alpha$, then $[\alpha] = \Theta_L$.

In the AGM paradigm knowledge sets [4] are taken to be theories, and informational changes are therefore regarded as transformations on theories. There are three principal types of AGM transformations; expansion, contraction and revision. These transformations allow us to model changes of information based on the principle of minimal change. Expansion is the simplest change, it models the incorporation of a formula. More formally, the expansion of a theory $T$ with respect to a formula $\alpha$, denoted as $T^+\alpha$, is defined to be the logical closure of $T$ and $\alpha$, that is $T^+\alpha = \text{Cn}(T \cup \{ \alpha \})$.

A contraction of $T$ with respect to $\alpha$, $T^\alpha$, involves the removal of a set of formulae from $T$ so that $\alpha$ is no longer implied. Formally, a well-behaved contraction operator $\gamma$ is any function from $K_L \times L$ to $K_L$, mapping $(T, \alpha)$ to $T^\alpha$, which satisfies the postulates (1) – (9), below. We define a very well-behaved contraction operator $\tilde{\gamma}$ to be a well-behaved contraction operator that satisfies the postulate (10), below.

\begin{align*}
(1) & \text{ For any } \alpha \in L \text{ and any } T \in K_L, T^\alpha \in K_L \\
(2) & T^\alpha \subseteq T \\
(3) & \text{ If } \alpha \notin T \text{ then } T^\alpha = T \\
(4) & \text{ If } \not\vdash \alpha \text{ then } \alpha \notin T^\alpha \\
(5) & T \subseteq (T^\alpha)^+ \\
(6) & \text{ If } \vdash \alpha \equiv \beta \text{ then } T^\alpha = T^\beta \\
(7) & T^\alpha \cap T^\beta \subseteq T_{\alpha \land \beta}^\alpha \\
(8) & \text{ If } \alpha \notin T_{\alpha \land \beta} \text{ then } T_{\alpha \land \beta} \subseteq T^\alpha \\
(9) & \text{ For every nonempty set } \Gamma \text{ of nontautological formulae, there exists a formula } \alpha \in \Gamma \text{ such that } \alpha \notin T_{\alpha \land \beta} \text{ for every } \beta \in \Gamma. \\
(10) & \text{ For every nonempty set } \Gamma \text{ of nontautological formulae, there exists a formula } \alpha \in \Gamma \text{ such that } \beta \notin T_{\alpha \land \beta} \text{ for every } \beta \in \Gamma. 
\end{align*}

A revision attempts to transform a theory as “little as possible” in order to incorporate a formula. Formally, a well-behaved revision operator $\ast$ is any function from
The smallest ordinal are the most plausible.

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disbelief

functions by $0$. We denote the family of all ordinal conditional functions by

with elements of $\Theta$

theories, into the class of ordinals such that there is some element of $\Theta$

function

operator that satisfies the postulate (10), below.

1) For any $\alpha \in \mathbb{L}$ and any $T \in \mathcal{K}_L$, $T^*_\alpha \in \mathcal{K}_L$

2) $\alpha \in T^*_\alpha$

3) $T^*_\alpha \subseteq T^*_\alpha$

4) If $\neg \alpha \notin T$ then $T^*_\alpha \subseteq T^*_\alpha$

5) $T^*_\alpha = \bot$ if and only if $\vdash \neg \alpha$

6) If $\vdash \alpha \equiv \beta$ then $T^*_\alpha = T^*_\beta$

7) $T^*_{\alpha \land \beta} \subseteq (T^*_\alpha)_{\beta}^+$

8) If $\neg \beta \notin T^*_\alpha$ then $(T^*_\alpha)_{\beta}^+ \subseteq T^*_{\alpha \land \beta}$

9) For every nonempty set $\Gamma$ of nontautological formulae, there exists a formula $\alpha \in \Gamma$ such that $\alpha \notin T^*_\alpha \forall \neg \beta$ for every $\beta \in \Gamma$.

10) For every nonempty set $\Gamma$ of nontautological formulae, there exists a formula $\alpha \in \Gamma$ such that $\beta \notin T^*_\alpha \forall \neg \beta$ for every $\beta \in \Gamma$.

The class of well-behaved revision operators was identified by Peppas [10] and shown to be a proper subclass of the class of revision operators satisfying (1) – (8). We note that (9) is not identical with the well-behaved postulate in [10], however the same family of revision operators is obtained.

In the next section we review Spohn’s ordinal conditional functions [18] and describe definitions from [18] and Gärdenfors [2].

3 ORDINAL CONDITIONAL FUNCTIONS

Spohn [18] represents a knowledge system as an ordinal conditional function which is defined over possible worlds. We represent possible worlds by consistent complete theories, in [22] we give an analogous analysis based on models as the underlying representation. An ordinal conditional function [18] defines a ranking of its domain which provides a ‘response schema for all possible consistent information’ [18]. More formally, we have the following definition [18,2].

Definition: An ordinal conditional function, OCF, is a function $C$ from $\Theta_L$, the set of all consistent complete theories, into the class of ordinals such that there is some element of $\Theta_L$ assigned the smallest ordinal 0. We denote the family of all ordinal conditional functions by $C$. If $C$ has a finite range, then we say $C$ is finite.

Intuitively $C \in C$ represents not only a well-ordering, but a plausibility grading of possible worlds [2] or a grading of disbelief [18], the worlds that are assigned the smallest ordinal are the most plausible.

Definition: The ordinal assigned to a nonempty set of consistent complete theories $\Delta \subseteq \Theta_L$ by an OCF is the smallest ordinal assigned to the elements of $\Delta$. That is, for $C \in C$ we have; $C(\Delta) = \min \{ C(K) : K \in \Delta \}$.

Definition: We define the knowledge set represented by $C \in C$ to be $\text{ks}(C) = \bigcap \{ K \in \Theta_L : C(K) = 0 \}$.

It is not hard to see that, the knowledge set represented by $C \in C$, $\text{ks}(C)$, is always a consistent theory.

Definition: Given a $C \in C$, for any nontautological formula $\alpha$, we say $\alpha$ is accepted with firmness $C([\neg \alpha])$, and call $C([\neg \alpha])$ the degree of acceptance of $\alpha$. A formula $\alpha$ is accepted if and only if $\alpha \in \text{ks}(C)$. If $\alpha$ and $\beta$ are both accepted then $\alpha$ is more firmly accepted than $\beta$ if and only if either $C([\neg \alpha]) > C([\neg \beta])$, or $\vdash \alpha$ and $\vdash \beta$. More generally, for nontautological formulae $\alpha$ and $\beta$ not necessarily in $\text{ks}(C)$, $\alpha$ is more plausible than $\beta$ if and only if either $C([\neg \alpha]) > C([\neg \beta])$, or $C([\beta]) > C([\alpha]) [2]$.

The tautologies can be thought of as being assigned to an ordinal greater than all ordinals in the range of $C$ (since $L$, and consequently the range of $C$, is countable such an ordinal exists). In [19] Spohn introduces a natural conditional function in which consistent complete theories are assigned a natural number rather than an ordinal, and $C(\emptyset) = \omega$, hence the degree of acceptance of the tautologies is $\omega$.

3.1 DYNAMICS OF ORDINAL CONDITIONAL FUNCTIONS

Revision and contraction operators take knowledge sets to knowledge sets. In this section we will see that transmutations take knowledge systems to knowledge systems, where a knowledge system is composed of a knowledge set together with a preference relation.

The relationship between Spohn’s conditionalization and the AGM paradigm is identified and discussed by Gärdenfors in [2], and it is this relationship that underpins and motivates our approach.

We begin our discussion with a representation result for revision. Theorem 1, below, provides a condition which characterizes a well-behaved and a very well-behaved revision operator, using an OCF and a finite OCF, respectively. Theorems 1 and 2 are based on the work of Grove [6] and Peppas [10].

Theorem 1: Let $T$ be a consistent theory of $L$. For every well-behaved (very well-behaved) revision operator $* \alpha$ for $T$, there exists a (finite) $C \in C$ such that $\text{ks}(C) = T$, and the condition below, henceforth
referred to as \((C^*)\), is true for every \(\alpha \in L\). Conversely, for every (finite) \(C \in C\) there exists a well-behaved (very well-behaved) revision operator \(*\) for \(ks(C)\) such that the \((C^*)\) condition is true for every \(\alpha \in L\).

\[
(ks(C))^*_\alpha = \begin{cases} \bigcap\{K \in [\alpha] : C(K) = C([\alpha])\} & \text{if } \neg \alpha \\ \bot & \text{otherwise} \end{cases}
\]

Similarly, Theorem 2, below, provides a condition which captures a well-behaved and a very well-behaved contraction operator using an OCF and a finite OCF, respectively.

**Theorem 2:** Let \(T\) be a consistent theory of \(L\). For every well-behaved (very well-behaved) contraction operator \(-\) for \(T\), there exists a (finite) \(C \in C\) such that \(ks(C) = T\), and the condition below, henceforth referred to as \((C^\bot)\), is true for every \(\alpha \in L\). Conversely, for every (finite) \(C \in C\) there exists a well-behaved (very well-behaved) contraction operator \(-\) for \(ks(C)\) such that the \((C^\bot)\) condition is true for every \(\alpha \in L\).

\[
(ks(C))^\bot_\alpha = \bigcap\{K \in \Theta_L : \text{either } C(K) = 0 \text{ or } K \in [\neg \alpha] \text{ with } C(K) = C([\neg \alpha])\}
\]

For contraction and revision the informational input is a formula \(\alpha\). We now define a transmutation of OCF’s where the informational input is composed of an ordered pair, \((\Delta, i)\), that is, a nonempty set of consistent complete theories (worlds) \(\Delta\) and a degree of acceptance \(i\). The interpretation [2] of this is that \(\Delta\) is the information to be accepted by the knowledge system, and \(i\) is the degree of acceptance with which this information is incorporated into the transmuted knowledge system. Note, for \(\Delta \subseteq \Theta_L\) we define \(\Delta^\prime\) to be the complement of \(\Delta\), that is, \(\Delta = \Theta_L \setminus \Delta^\prime\).

**Definition:**
We define a transmutation schema for OCF’s, \(*\), to be an operator from \(C \times \{2^{\Theta_L} \setminus \emptyset, \Theta_L\} \times O\) to \(C\), where \(O\) is an ordinal, such that \((C, \Delta, i) \mapsto C^*(\Delta, i)\) which satisfies:

(i) \(C^*(\Delta, i)(\Delta^\prime) = i\), and

(ii) \(ks(C^*(\Delta, i)) = \begin{cases} \bigcap\{K \in \Delta : C(K) = C(\Delta)\} & \text{if } i \geq 0 \\ \bigcap\{K \in \Theta_L : \text{either } C(K) = 0, \text{ or } K \in \Delta^\prime \text{ with } C(K) = C(\Delta^\prime)\} & \text{otherwise} \end{cases}\)

We say \(C^*(\Delta, i)\) is a \((\Delta, i)\)-transmutation of \(C\). The definition excludes a transmutation with respect to an empty set of worlds, hence a contradiction is not acceptable information.

It is not hard to see that, for \(i \geq 0\) and a nontautological consistent formula \(\alpha\), an \(([\alpha], i)\)-transmutation of \(C\) is an OCF in which the ‘smallest’ worlds in \([\alpha]\) are mapped to zero and the ‘smallest’ world in \([\neg \alpha]\) is mapped to \(i\). Hence the knowledge set represented by the transmuted knowledge system is equivalent to the revised knowledge set \((ks(C))^*_\alpha\), via \((C^*)\). Similarly, in view of Theorem 2, we have that, a contraction of \((ks(C))^\bot_\alpha\) can be modeled by an \(([\alpha], 0)\)-transmutation of \(C\), where the ‘smallest’ \([\neg \alpha]\) worlds are assigned zero, and hence \([\alpha]\) is no longer accepted, in other words, \([\alpha]\) is ‘neutralized’ [18].

Rumelhart and Norman [14] distinguish three modes of learning: accretion, tuning, and restructuring. Accretion involves the expansion of episodic memory without changing semantic memory, and tuning involves revision of semantic memory in order to accommodate new information. Restructuring involves major reorganization of semantic memory. According to Sowa [17, p331] restructuring takes place when a knowledge system attains new insight, and a ‘revolution’ takes place that repackages old information. In the following definition we define a restructuring of an OCF, as a reorganization of accepted information.

**Definition:** Let \(C\) be an OCF, and let \(\Delta\) be a nonempty, proper subset of \(\Theta_L\) such that \(C(\Delta) > 0\). If \(i\) is a nonzero ordinal, then we refer to a \((\Delta, i)\)-transmutation of \(C\) as a **restructuring**.

A restructuring is discussed by both Spohn [18] and Gärdenfors [2]. In our terminology whenever a \((\Delta, i)\)-transmutation of \(C\) is a restructuring, if \(i > C(\Delta) > 0\), then the knowledge system has additional reason for accepting \(\Delta\), and its firmness is increased, that is, strengthened. On the other hand, if \(C(\Delta) > i > 0\), then the knowledge system has reduced reason for accepting \(\Delta\), and its firmness is decreased, that is, weakened.

The imposing problem for iterated revision is; how are the worlds ordered after a revision, or a contraction. Spohn [18] considers that it might be that after information \(\Delta\) is accepted, all possible worlds in \(\Delta\) are less disbelieved than worlds in \(\Delta^\prime\). The result of such a transmutation would be that the knowledge system has overwhelming confidence in \(\Delta\), however knowledge systems must also be capable of accepting information with less confidence.

Spohn [18] has argued that conditionalization, defined below, is a desirable transmutation, for instance it is reversible (that is, there is an inverse conditionalization) and commutative. Intuitively, conditionalization
means that becoming informed about $\Delta$, a proper and nonempty subset of $\Theta_L$, does not change the grading of disbelief restricted to either $\Delta$ or $\bar{\Delta}$, rather the worlds in $\Delta$ and $\bar{\Delta}$ are shifted in relation to one another [2], and this Spohn argues is reasonable, since becoming informed only about $\Delta$ should not change $C$ restricted to $\Delta$, or $C$ restricted to $\bar{\Delta}$. The construction in Theorem 3 is based on left-sided subtraction of ordinals that is, for $i \leq j$; $-i + j$ is the uniquely determined ordinal $k$ such that $i + k = j$.

**Theorem 3:** For $\Delta \subset \Theta_L$, $\Delta \neq \emptyset$, and $i$ an ordinal, $C^*(\Delta, i)$, defined below, is a $(\Delta, i)$-transmutation of $C$. We refer to this transmutation as the $(\Delta, i)$-conditionalization of $C$.

\[
C^*(\Delta, i)(K) = \begin{cases} 
-C(\Delta) + C(K) & \text{if } K \in \Delta \\
-C(\bar{\Delta}) + C(K) + i & \text{otherwise}
\end{cases}
\]

The $(\Delta, i)$-conditionalization of $C$ is the combination of $C$ restricted to $\Delta$ left unaltered, and $C$ restricted to $\bar{\Delta}$ shifted up $i$ grades [18].

A property enjoyed by conditionalization is that any transmutation of $C$, where the underlying language is finitary, can be achieved by a finite sequence of conditionalizations. This is seen by observing, that for a $K \in \Theta_L$ such that $C(K) > 0$, $C^*(\{K\}, i)$ assigns the consistent complete theory $K$ the ordinal $i$ and assigns $C(K')$ to each of the worlds $K' \in \{K\}$.

We now explore another transmutation, an adjustment, in which only the least disbelieved worlds containing the information the knowledge system is accepting are reassigned zero. Intuitively, an adjustment is a transmutation which is commanded by the principle of minimal change, that is, an OCF is changed or disturbed ‘as little as necessary’ so as to accept the information with the desired degree of acceptance. In other words, as much structure of the OCF persists after the adjustment as possible.

**Theorem 4:** For $\Delta \subset \Theta_L$, $\Delta \neq \emptyset$, and $i$ an ordinal, $C^*(\Delta, i)$, where $^*$ is defined below, is a $(\Delta, i)$-transmutation of $C$. We refer to this transmutation as the $(\Delta, i)$-adjustment of $C$.

\[
C^*(\Delta, i) = \begin{cases} 
C^-(\Delta) & \text{if } i = 0 \\
(C^-(\Delta))^*(\Delta, i) & \text{if } 0 < i < C(\bar{\Delta}) \\
C^x(\Delta, i) & \text{otherwise}
\end{cases}
\]

where

\[
C^-(\Delta)(K) = \begin{cases} 
0 & \text{if } K \in \Delta \text{ and } C(K) = C(\Delta) \\
C(K) & \text{otherwise}
\end{cases}
\]

Essentially, $C^-(\Delta)$ models the contraction of $\Delta$, and is used for $(\Delta, i)$-transmutations where $\Delta$ is accepted, and $C(\Delta) > i$, that is, a restructuring where $\Delta$’s firmness is decreased.

We are not advocating that adjustment is a more desirable transmutation than conditionalization, however in [21] we have shown that it lends itself to theory base transmutations. It can also be shown that in a finitary language, we can find a sequence of adjustments which will result in an arbitrary transmutation. For instance, an adjustment can be used to reassign all but one consistent complete theory, say $K_1 \in \Theta_L$ where $C(\{K_1\}) = 0$ to the largest desired ordinal, say $k$, by $C^*\{\{K_1\}, k\}$. Then grade by grade the remainder $\{K_1\}$ can be assigned the desired ordinal, for instance $(C^*\{\{K_1\}, k\})^*\{\{K_1 \cup K_2\}, j\}$ assigns $K_1$ to $0$, $K_2$ to $k$, and $\Theta_L \setminus \{K_1 \cup K_2\}$ to $j$. Continuing in this way, $(C^*\{\{K_1\}, k\})^*\{\{K_1 \cup K_2 \cup K_3\}, i\}$ assigns $K_1$ to $0$, $K_2$ to $K_3$, $K_3$ to $j$, and $\Theta_L \setminus \{K_1 \cup K_2 \cup K_3\}$ to $i$. This process can be continued until all $K_i \in \Theta_L$ are assigned their desired ordinal.

As noted earlier any transmutation where the underlying language is finitary can be achieved by a finite sequence of conditionalizations, therefore a sequence of conditionalizations can be used to effect an adjustment, and conversely.

One view of the difference between conditionalization and adjustment is that in order to accomodate the desired informational change, conditionalization preserves the relative gradings of $C$ restricted to $\Delta$ and $\bar{\Delta}$, whilst adjustment minimizes changes to the absolute gradings $C$ as a whole. Adjustments are discussed further in Section 6.
4 ORDINAL EPISTEMIC ENTRENCHMENT FUNCTIONS

Intuitively, an ordinal epistemic entrenchment function maps the formulae in a language to the ordinals, the higher the ordinal assigned the more firmly it is held. Throughout the remainder of this paper it will be understood that $\mathcal{O}$ is an ordinal chosen to be sufficiently large for the purpose of the discussion. We now formally define an ordinal epistemic entrenchment function.

**Definition:** An ordinal epistemic entrenchment function, OEF, is a function $E$ from the formulae in $L$ into the class of ordinals such that the following conditions are satisfied.

(OEF1) For all $\alpha, \beta \in L$, if $\alpha \vdash \beta$, then $E(\alpha) \leq E(\beta)$.

(OEF2) For all $\alpha, \beta \in L$, $E(\alpha) \leq E(\alpha \land \beta)$ or $E(\beta) \leq E(\alpha \land \beta)$.

(OEF3) $E(\alpha)$ if and only if $E(\alpha) = 0$.

(OEF4) If $\alpha$ is inconsistent, then $E(\alpha) = 0$.

Intuitively, $E$ represents an epistemic entrenchment [2,4] grading of formulae; the higher the ordinal assigned to a formula the more entrenched that formula is. Whenever the codomain of $E$ is ordinal, Spohn’s natural conditional functions [19], then we can, and will, take $E(\alpha) = \omega$ for all $\alpha$. If $E$ has a finite range, then we say $E$ is finite.

**Definition:**

We denote the family of all ordinal epistemic entrenchment functions to be $E$. The knowledge set represented by $E \in E$ is $ks(E) = \{\alpha \in L : E(\alpha) > 0\}$.

**Definition:** Given an $E \in E$, for any formula $\alpha$ we say $\alpha$ is accepted with firmness $E(\alpha)$, and call $E(\alpha)$ the degree of acceptance of $\alpha$. A formula $\alpha$ is accepted if and only if $\alpha \in ks(E)$.

A knowledge set, $ks(E)$, is the set of accepted formulae and is always consistent. Gärdenfors and Makinson [4] have shown that for a finitary language an epistemic entrenchment ordering is determined by its dual atoms. Similarly, we can describe an OEF for a finite language by assigning an ordinal to each dual atom. The ordinal assigned to all other formulae in $L$ is then uniquely determined by (OEF1) – (OEF4).

4.1 DYNAMICS OF ORDINAL EPISTEMIC ENTRENCHMENT FUNCTIONS

In this section we discuss the dynamics of ordinal epistemic entrenchment functions, in particular we discuss their revisions, contractions, and transmutations.

In Theorems 5 and 6 we establish several conditions which characterize well-behaved and very well-behaved revision and contraction operators, using OEF’s and finite OEF’s, respectively. These results are based on the work of Gärdenfors and Makinson in [4].

**Theorem 5:** Let $T$ be a consistent theory of $L$. For every well-behaved (very well-behaved) revision operator $*$ for $T$, there exists an (finite) $E \in E$, such that $ks(E) = T$ and the condition below, henceforth referred to as $(E^*)$, is true for every $\alpha \in L$. Conversely, for every (finite) $E \in E$ there exists a well-behaved (very well-behaved) revision operator $*$ for $ks(E)$ such that the $(E^*)$ condition is true for every $\alpha \in L$.

**Theorem 6:** Let $T$ be a consistent theory of $L$. For every well-behaved (very well-behaved) contraction operator $-$ for $T$, there exists an (finite) $E \in E$ such that $ks(E) = T$, and the condition below, henceforth referred to as $(E^-)$, is true for every $\alpha \in L$. Conversely, for every (finite) $E \in E$ there exists a well-behaved (very well-behaved) contraction operator $-$ for $ks(E)$ such that the $(E^-)$ condition is true for every $\alpha \in L$.

The conditions $(E^*)$, and $(E^-)$ determine a new knowledge set when a formula $\alpha$ is incorporated or removed, however these conditions do not determine another OEF, upon which a subsequent revision, or contraction could be specified. According to Rott [12] it is not theories that have to be revised but epistemic entrenchment orderings. The definition below describes a transmutation of an OEF into another OEF, such that a nontautological consistent formula $\alpha$ is accepted with degree $i$.

**Definition:** We define a transmutation schema for OEF’s, $*$, to be an operator from $E \times L^\omega \times O$ to $E$, where $L^\omega$ is the set of consistent nontautological formulae in $L$, and $i < O$, such that $(E, \alpha, i) \mapsto E^*(\alpha, i)$ satisfies:

(i) $E^*(\alpha, i)(\alpha) = i$, and
(ii) \( \text{ks}(E^*(\alpha, i)) = \begin{cases} \{ \beta \in L : E(\neg \alpha) < E(\neg \alpha \lor \beta) \} & \text{if } i > 0 \\
\{ \beta \in \text{ks}(E) : E(\alpha) < E(\alpha \lor \beta) \} & \text{otherwise} \end{cases} \)

We say \( E^*(\alpha, i) \) is an \((\alpha, i)\)-transmutation of \( E \). A transmutation is not defined with respect to tautological, or inconsistent formulae. An OEF is incapable of representing an inconsistent knowledge set, therefore we should not expect a transmutation of a knowledge system to accept inconsistent information.

Intuitively, for a nontautological consistent formula \( \alpha \), a transmutation \( E^*(\alpha, i) \), is an OEF where \( \text{ks}(E^*(\alpha, i)) \) represents a ‘minimal change’ of the knowledge set represented by \( E \), that is \( \text{ks}(E) \), and \( \alpha \) is assigned the degree of acceptance \( i \).

In the following definition we define a restructuring of an OEF, as a reorganization of accepted information.

**Definition:** Let \( \alpha \in L^\infty \), and let \( E \) be an OEF, where \( \alpha \) is accepted, that is, \( E(\alpha) > 0 \). If \( i \) is an ordinal such that \( 0 < i < \mathcal{O} \), then we refer to a \((\alpha, i)\)-transmutation of \( E \) as a **restructuring**.

Theorems 7 and 8, below, establish that conditionalization and adjustment of OEF’s, respectively, are transmutations.

**Theorem 7:** Let \( L \) be a finitary language, let \( E \in \mathcal{E} \), let \( i \) be an ordinal such that \( i < \mathcal{O} \), and let \( \alpha \in L^\infty \). Then \( E^*(\alpha, i) \) defined below, is an \((\alpha, i)\)-transmutation of \( E \). We refer to this transmutation as the \((\alpha, i)\)-conditionalization of \( E \).

\[
E^*(\alpha, i)(\beta) = \begin{cases} -E(\neg \alpha) + E(\beta) & \text{if } \alpha \land \neg \beta \not\vdash \bot \\
-E(\alpha) + E(\beta) + i & \text{otherwise,} \\
\end{cases}
\]

where \( \beta \in L \) is a dual atom.

**Theorem 8:** Let \( E \in \mathcal{E} \), let \( i \) be an ordinal such that \( i < \mathcal{O} \), and let \( \alpha \in L^\infty \). Then \( E^*(\alpha, i) \), where * is defined below, is an \((\alpha, i)\)-transmutation of \( E \). We refer to this transmutation as the \((\alpha, i)\)-adjustment of \( E \).

\[
E^*(\alpha, i) = \begin{cases} E^-(\alpha) & \text{if } i = 0 \\
(\alpha \land \neg \beta \not\vdash \bot) & \text{if } 0 < i < E(\alpha) \\
E^*(\alpha, i) & \text{otherwise} \\
\end{cases}
\]

where,

Both conditionalizations and adjustments are transmutations of OEF’s, however unlike a conditionalization, an adjustment does not refer to the dual atoms in the theory and is therefore more general in that it can be used on arbitrary languages rather than just finitary ones. Moreover, adjustments have been shown [21] to be very easily adapted to theory base transmutations based on encomencments [20].

We note however, for a finitary language, as in the case of an OCF, an individual dual atom can be assigned a firmness, say \( i \), without changing the firmness of any other dual atom using both conditionalization and adjustment. Therefore it can be shown that any transmutation of an OEF in a finitary language, can be achieved by a finite sequence of conditionalizations or adjustments.

Intuitively, an \((\alpha, i)\)-adjustment of \( E \) is a transmutation that minimizes the changes to \( E \) so that a formula \( \alpha \) is accepted with firmness \( i \). Computationally, the process of adjustment is straightforward [20], and whenever the change is not maxichoice [1,2] less explicit information is required than is the case for conditionalization.

Adjustments \( E^*(\alpha, i) \) possess an interesting property in that, all the accepted formulae of \( E \) which are held more firmly than \( \max\{i, E(\neg \alpha), E(\alpha)\} \) are retained and not reassigned a different ordinal during the transmutation *. This behaviour is intuitively appealing, since we would not expect a transmutation which affects weakly held information such as a goldfish is blocking the main pump, to change the degree of acceptance of more firmly held information, such as if main pump is blocked, then the temperature will rise.
The ordinal functions \( \mathbf{C} \) and \( \mathbf{E} \) are similar, then their transmuted knowledge sets are equivalent.

**Theorem 9**: Let \( \alpha \in \mathbb{L}^\omega \) and let \( i \) be an ordinal such that \( i < \mathcal{O} \). Let \( * \) be a transmutation schema for OCF's. Let \( * \) be a transmutation schema for OEF's. For \( \mathbf{E} \in \mathcal{E} \) and \( \mathbf{C} \in \mathcal{C} \), \( \text{ks}(\mathbf{E}^*(\alpha, i)) = \text{ks}(\mathbf{C}^*((\alpha], i)) \) for all \( i < \mathcal{O} \) if and only if \( \mathbf{C} \) and \( \mathbf{E} \) are similar.

The definition below describes when an OCF and an OEF are equivalent.

**Definition**: We define \( \mathbf{C} \in \mathcal{C} \) and \( \mathbf{E} \in \mathcal{E} \) to be equivalent if and only if they satisfy the following condition for all nontautological formulae \( \alpha \).

\[
(\mathbf{EC}) \quad \mathbf{E}(\alpha) = \mathbf{C}([-\alpha]).
\]

Intuitively, \( \mathbf{E} \) and \( \mathbf{C} \) are equivalent if and only if all nontautological formulae possesses precisely the same degree of acceptance with respect to both \( \mathbf{E} \) and \( \mathbf{C} \). Clearly, if \( \mathbf{C} \) and \( \mathbf{E} \) are equivalent then they are similar. Hence we obtain the following corollary, which says that, if the ordinal functions \( \mathbf{C} \) and \( \mathbf{E} \) are equivalent, then their transmuted knowledge sets are equivalent.

**Corollary 10**: Let \( \alpha \in \mathbb{L}^\omega \) and let \( i \) be an ordinal such that \( i < \mathcal{O} \). Let \( * \) be a transmutation schema for OCF's. Let \( * \) be a transmutation schema for OEF's. If \( (\mathbf{EC}) \) holds for \( \mathbf{E} \in \mathcal{E} \) and \( \mathbf{C} \in \mathcal{C} \), then \( \text{ks}(\mathbf{E}^*(\alpha, i)) = \text{ks}(\mathbf{C}^*((\alpha], i)). \)

The definition below describes when transmutations for an OCF and an OEF are equivalent.

**Definition**: Given a transmutation schema \( * \) on OCF's, and a transmutation schema \( * \) on OEF's, we define \( \mathbf{C} \in \mathcal{C} \) and \( \mathbf{E} \in \mathcal{E} \) to be equivalent with respect to \( * \) and \( * \) if and only if they satisfy the following condition for all nontautological, consistent formulae \( \alpha \) and \( \beta \), and all ordinals \( i < \mathcal{O} \).

\[
(\mathbf{E}'\mathbf{C}') \quad \mathbf{E}^*((\alpha, i)\beta) = \mathbf{C}^*((\alpha], i)[(-\beta)]).
\]

Intuitively, an OEF, \( \mathbf{E} \), and an OCF, \( \mathbf{C} \), are equivalent with respect to \( * \) and \( * \) if and only if all nontautological formulae possess exactly the same degree of acceptance in the ordinal functions, \( \mathbf{E}^*(\alpha, i) \) and \( \mathbf{C}^*((\alpha], i) \), after every possible transmutation.

In the following theorems we show if \( \mathbf{E} \) and \( \mathbf{C} \) are equivalent then both their conditionalizations and their adjustments are equivalent, that is, \( (\mathbf{EC}) \) implies \( (\mathbf{E}'\mathbf{C}') \), and conversely.

**Theorem 11**: Let \( \mathbb{L} \) be a finitary language. Let \( \mathbf{C} \in \mathcal{C} \), and \( \mathbf{E} \in \mathcal{E} \). Given a conditionalization \( * \) on OCF's, and a conditionalization \( * \) on OEF's, \( (\mathbf{EC}) \) holds for...
C and E if and only if (E' C') holds for E'((α, i) and C''((α, i)), for all nontautological consistent formulae α, and all ordinals i < O.

Theorem 12: Let C ∈ C, and E ∈ E. Given an adjustment * on OCF’s, and an adjustment * on OEF’s, (EC) holds for C and E if and only if (E' C') holds for E'(α, i) and C''((α, i)), for all nontautological consistent formulae α, and all ordinals i < O.

6 RELATED WORK

Various approaches to the iterated revision problem other than Spohn’s have been explored, we briefly compare and contrast some of them. Schlechta [15] describes a preference relation on L, from which an epistemic entrenchment can be derived for each knowledge set, he also provides a means of naturally constructing this preference relation from a probability distribution. Hansson [8], using what he calls superselectors, develops a more general account of the AGM paradigm. A superselector can be construed to be a function that assigns a selection function [1,2], to sets of formulae or theory bases. Rott uses a generalized epistemic entrenchment [13], from which a family of revision and contraction operators can be derived, one for every theory in the language.

These approaches, with the exception of Hansson’s, associate a preference relation with a theory which means that the dynamical behaviour of a given theory is fixed. Therefore an informational restructure is not naturally supported. The fundamental reason for this is that in contrast to Spohn’s approach, a formula is the only informational input, consequently the resulting knowledge set is constructed regardless of the evidential strength of the informational input.

Other forms of informational input have been used, in particular, Spohn [18] has described the conditionalization of an OCF by another OCF which embodies the new evidence, and Nayak [9] describes a mechanism for incorporating an epistemic entrenchment ordering representing new evidence, into another epistemic entrenchment ordering.

Spohn [18] has observed that OCF’s are related to degrees of potential surprize [16], and in view of the translation conditions described in Section 5 so too are OEF’s. According to Shackle [16, p80] surprize, is a function from a field of propositions into a closed interval [0, 1] such that for all propositions Δ, Ψ ∈ 2^Θ, the following are satisfied:

• surprize(∅) = 1.
• either surprize(Δ) = 0, or surprize(Δ) = 0.
• surprize(Δ ∪ Ψ) = min({surprize(Δ), surprize(Ψ)}).

The maximal degree of potential surprize is 1. The major difference with OCF’s for instance is that there is no need for a maximal degree of firmness, but recall that Spohn later used a natural conditional function whose maximal degree of firmness is ω. Spohn notes that Shackle does not present a transmutation schema for potential surprize. However it is not hard to see that the transmutations of OCF’s could be used for such a purpose.

Transmutations have been used [24] to support Spohn’s notion of ‘reason for’, which can be specified [18, 3] by:

α is a reason for β if and only if raising the epistemic rank of α would raise the epistemic rank of β.

Clearly, this condition can be expressed using the degree of acceptance, such a notion of ‘reason for’ will be dependent on the particular transmutation employed. In [24] Williams et al. have provided a simple condition which captures ‘reason for’ when adjustments are used as the underlying transmutation. Furthermore, they use adjustments to determine the relative plausibility of alternative explanations, based on the principle that one would expect a best explanation to be one that increases the degree of acceptance of the expandum at least as much as any other explanation.

Transmutations for theory bases based on ensconce-ments [20] have been investigated by Williams in [21]. In particular, a partial OEF is used to specify a theory base and its associated preference relation. A partial OEF can be used to implicitly capture an OEF, on which theory transmutations could be used. Alternatively, one can specify transmutations for them using only the explicit information they represent. Both of these approaches are used in [21]. It turns out that adjustments are straightforward transmutations for theory bases. In addition, Williams [23] recasts Spohn’s ‘reason for’ in a theory base setting by using adjustments of partial OEF’s.

7 DISCUSSION

We have explored transmutations of knowledge systems, by considering not only how the knowledge set is revised, but how the underlying preference structure for the knowledge system is revised, or more precisely transmuted.

We have provided representation results for well-behaved and very well-behaved theory change operators. In particular, transmuted knowledge sets of OCF’s, and OEF’s, characterize well-behaved con-
traction, and well-behaved revision within the AGM paradigm. Furthermore, transmutations of finite OCF’s, and finite OEF’s, characterize very well-behaved contraction, and very well-behaved revision operators. For a finitary language all OCF’s and OEF’s will be finite and consequently the transmutations representing revision and contraction will be very well-behaved.

We have provided explicit and perspicuous conditions which capture the relationship between OCF’s, and OEF’s. These conditions can be used to construct each of these structures from the other, such that the knowledge set and its dynamical properties are preserved.

We have also provided an explicit condition which relates the transmutations on OCF’s, and OEF’s, and moreover we showed that both conditionalizations and adjustments satisfy this condition.

Since any transmutation of a knowledge system in a finitary language can be achieved by a sequence of conditionalizations or adjustments of OCF’s, or OEF’s, they provide a powerful mechanism for supporting the computer-based implementation of a knowledge system transmutation. With judicious modularization of suitable applications parallel processing could be adopted, since the reassignment of ordinals for each consistent complete theory (or dual atom) is determined by its compatibility with the new information, and is completely independent of the compatibility with other consistent complete theories (or dual atoms).

We would expect that for a given application, domain constraints could be used to identify consistent complete theories representing possible world states which are so inconceivable as to always be assigned very remote grades or ‘large’ ordinals.

If the underlying preference structure of a knowledge system is an OCF then model checking [6] can be used to implement transmutations. Alternatively, if the underlying preference structure is an OEF, then theorem proving techniques can be used to support the implementation.

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References

13. Rott, H., On the logic of theory change: Partial meet contraction and prioritized base contraction, Report


